Standing surface waves of finite amplitude

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Gravity waves on the surface of an inviscid incompressible fluid of finite depth are considered. The waves are assumed to be periodic in time and in the horizontal direction. The surface profile, potential function, pressure and frequency of the motion are determined (to third order) as series in powers of the amplitude divided by the wavelength. It is found that the frequency increases with amplitude for depths less than a certain multiple of the wavelength and decreases with increasing amplitude for greater depths. Graphs of the surface profile and of the pressure as a function of depth are included.

1. Introduction

Gravity waves on the surface of a liquid are governed by non-linear equations. In the classical theory of such waves, the equations are linearized. Thus, the results of that theory yield the linear terms in the expansion of the wave motion in powers of the amplitude. We shall determine two additional terms in this expansion for standing waves in a liquid of uniform finite depth. Our results will then exhibit various deviations from those of the linear theory. For example, the period will depend upon the amplitude, the motion will not be sinusoidal in time nor in space, and the maximum elevation will not equal the minimum depression. These effects become more pronounced the larger the amplitude. They are described and depicted graphically in § 4.

Similar results have been obtained for liquid of infinite depth by Penney & Price (1952). Our results for the wave profile and the period coincide with theirs when we permit the depth to become infinite. However, our analysis indicates that in the expression for the fluid pressure there are additional terms, independent of the vertical distance, which they have not taken into account. Airy, Stokes, Rayleigh and others have obtained analogous results for progressing waves.

We formulate the problem in the next section and solve it by a perturbation method in § 3. Then we discuss the results in § 4 where some graphs are also given. In the Appendix we show how the method of Penney & Price can be modified to yield the same results, although it requires more labour than our method.

2. Formulation

Let us consider the time-periodic irrotational two-dimensional motion of an inviscid incompressible fluid bounded below by a rigid horizontal bottom and above by a free surface. We suppose the motion to be periodic in the horizontal direction and symmetric about the vertical plane $x = 0$. Then we may confine our attention to the fluid between that plane and a parallel plane one half wavelength from it. Let λ denote the wavelength, $k = 2\pi/\lambda$ the propagation constant, $k^{-1}h$ the mean depth of the liquid, $k^{-1}x$ and $k^{-1}y$ the distances along the horizontal and vertical axes, respectively, g the acceleration of gravity, $(kg)^{\frac{1}{2}}\omega$ the angular frequency and $(kg)^{-\frac{1}{2}}\omega^{-1}t$ the time. In addition let a be a measure of the wave amplitude, the precise meaning of which will become clear later. Then we define $\epsilon = ka$ and let $\epsilon k^{-1}\eta(x,t)$ denote the elevation of the surface above the mean level given by the plane $y = 0$. Finally, we introduce the potential function $\epsilon g^{\frac{1}{2}}k^{-\frac{3}{2}}\phi(x,y,t)$.

In terms of the dimensionless quantities which we have just introduced, the equations which govern the motion are

$$\Delta\phi = 0 \quad \text{in} \quad 0 \leqslant x \leqslant \pi, \; -h \leqslant y \leqslant \epsilon\eta(x,t), \tag{1}$$

$$\eta + \omega\phi_t + \tfrac{1}{2}\epsilon(\phi_x^2 + \phi_y^2) = 0 \quad \text{on} \quad y = \epsilon\eta(x,t), \tag{2}$$

$$\phi_y = \omega\eta_t + \epsilon\phi_x\eta_x \quad \text{on} \quad y = \epsilon\eta(x,t), \tag{3}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad x = 0, \, x = \pi, \, y = -h, \tag{4}$$

$$\int_0^\pi \eta(x,t)\,dx = 0, \tag{5}$$

$$\nabla\phi(x,y,t+2\pi) = \nabla\phi(x,y,t), \tag{6}$$

$$\int_0^\pi\int_0^{2\pi} \eta(x,t)\cos t\,\cos x\,dt\,dx = 0, \tag{7}$$

$$\int_{-h}^0\int_0^\pi\int_0^{2\pi} \phi(x,y,t)\,\cos t\cos x\,dt\,dx\,dy = \tfrac{1}{2}\pi^2(\tanh h)^{\frac{1}{2}}. \tag{8}$$

Equation (1) expresses the incompressibility of the fluid; (2) is the condition that the pressure at the surface, as obtained from Bernoulli's equation, be constant; (3) is the condition that a particle on the surface remain on the surface; (4) asserts that the bottom $y = -h$ is rigid and that $x = 0$ and $x = \pi$ are planes of symmetry of the motion; (5) is the condition that the mean surface is $y = 0$; (6) asserts the periodicity of the velocity components; (7) and (8) fix the phase and amplitude of the motion. Equation (8) shows that in the dimensional units, a is proportional to the Fourier coefficient of $\eta(x,t)$ with respect to the linearized surface profile $\sin t\cos x$. In fact a is just the amplitude of the linearized surface wave motion.

The pressure $p(x,y,t)$ is given by Bernoulli's equation

$$\frac{k}{\rho g}(p - p_0) = -y - \epsilon\omega\phi_t - \tfrac{1}{2}\epsilon^2(\phi_x^2 + \phi_y^2). \tag{9}$$

Here p_0 denotes the pressure of the atmosphere above the fluid and ρ denotes the constant density of the fluid.

The problem we consider is that of determining the dimensionless surface profile $\eta(x,t)$, the dimensionless potential function $\phi(x,y,t)$ and the dimensionless angular frequency ω satisfying equations (1) to (8). We note that the solution also depends upon the dimensionless depth h, which is proportional to the actual depth divided by the wavelength, and the dimensionless constant $\epsilon = ka$, which is proportional to the linearized amplitude divided by the wavelength. We shall solve the problem by determining the first three terms in the expansion of the solution in powers of ϵ. Additional terms can be found by continuing our procedure.

We shall find that, for certain values of h, the problem we have formulated does not have a unique solution. This lack of uniqueness does not refer to the arbitrary constant which may be added to ϕ. These values of h are those for which the linear theory yields, for some harmonic (in space), a frequency (in time) which is an integral multiple of the fundamental frequency. According to the linear theory the frequency of the nth spatial harmonic is $(n \tanh nh)^{\frac{1}{2}}$. Thus we can make the problem have a unique solution by imposing the condition

$$\frac{n \tanh nh}{\tanh h} \neq (\text{integer})^2 \quad (n = 2, 3, \ldots). \tag{10}$$

Of course the solution we shall obtain will be a solution for any value of h, but it will not be the only solution satisfying (1) to (8) unless h satisfies (10).

3. Perturbation solution

We assume that η, ϕ and ω have limits η^0, ϕ^0 and ω_0 as ϵ tends to zero. If we set $\epsilon = 0$ in equations (1) to (8), we find that all but (2) and (3) are unchanged in form while these two become

$$\eta^0 + \omega_0 \phi_t^0 = 0 \quad \text{on} \quad y = 0, \tag{2^0}$$

$$\phi_y^0 - \omega_0 \eta_t^0 = 0 \quad \text{on} \quad y = 0. \tag{3^0}$$

The zero-order problem, with (2^0) and (3^0) instead of (2) and (3), is just the classical linear problem. It has the unique solution

$$\eta^0 = \sin t \cos x, \tag{11}$$

$$\phi^0 = \frac{\omega_0}{\sinh h} \cos t \cos x \cosh (y+h), \tag{12}$$

$$\omega_0^2 = \tanh h. \tag{13}$$

We now assume that η, ϕ and ω have derivatives with respect to ϵ at $\epsilon = 0$ and we denote them by η^1, ϕ^1 and ω_1. Then we differentiate equations (1) to (8) with respect to ϵ and let ϵ tend to zero. In differentiating (2) and (3) we utilize the relation

$$\frac{d}{d\epsilon} \phi(x, \epsilon\eta, t, \epsilon) = \left[\frac{\partial}{\partial \epsilon} + (\eta + \epsilon\eta_\epsilon) \frac{\partial}{\partial y}\right] \phi. \tag{14}$$

Upon differentiating (2) and (3), but before setting $\epsilon = 0$, we obtain

$$\eta_\epsilon + \omega_\epsilon \phi_t + \omega[\phi_{t\epsilon} + (\eta + \epsilon\eta_\epsilon) \phi_{ty}] = -\tfrac{1}{2}(\phi_x^2 + \phi_y^2) - \epsilon\phi_x[\phi_{x\epsilon} + (\eta + \epsilon\eta_\epsilon) \phi_{xy}] - \epsilon\phi_y[\phi_{y\epsilon} + (\eta + \epsilon\eta_\epsilon) \phi_{yy}] \quad \text{on} \quad y = \epsilon\eta(x,t), \tag{15}$$

$$\phi_{y\epsilon} + (\eta + \epsilon\eta_\epsilon) \phi_{yy} - \omega_\epsilon \eta_t - \omega\eta_{t\epsilon} = \eta_x \phi_x + \epsilon\eta_{x\epsilon} \phi_x + \epsilon\eta_x[\phi_{x\epsilon} + (\eta + \epsilon\eta_\epsilon) \phi_{xy}] \quad \text{on} \quad y = \epsilon\eta(x,t). \tag{16}$$

When we set $\epsilon = 0$ in the differentiated form of (1) and (4) to (7), these equations retain their form with the superscript 'one' affixed to η and ϕ. From (15), (16) and (8), we obtain

$$\eta^1 + \omega_0 \phi^1_t = -\tfrac{1}{2}[(\phi^0_x)^2 + (\phi^0_y)^2] - \omega_0 \eta^0 \phi^0_{ty} - \omega_1 \phi^0_t \quad \text{on} \quad y = 0, \tag{2^1}$$

$$\phi^1_y - \omega_0 \eta^1_t = \eta^0_x \phi^0_x - \eta^0 \phi^0_{yy} + \omega_1 \eta^0_t \quad \text{on} \quad y = 0, \tag{3^1}$$

$$\int_0^\pi \int_0^{2\pi} \eta^1(x,t) \sin t \cos x \, dt \, dx = 0. \tag{8^1}$$

To solve the differentiated equations for η^1, ϕ^1 and ω_1, we first insert into (2^1) and (3^1) the zero-order quantities given by (11) to (13). Upon doing this and simplifying the results, we obtain

$$\eta^1 + \omega_0 \phi^1_t = \tfrac{1}{8}[(\omega_0^2 - \omega_0^{-2}) + (\omega_0^2 + \omega_0^{-2}) \cos 2x - (3\omega_0^2 + \omega_0^{-2}) \cos 2t$$
$$- (3\omega_0^2 - \omega_0^{-2}) \cos 2t \cos 2x] + \frac{\omega_1}{\omega_0} \sin t \cos x \quad \text{on} \quad y = 0, \tag{17}$$

$$\phi^1_y - \omega_0 \eta^1_t = -\frac{1}{2\omega_0} \sin 2t \cos 2x + \omega_1 \cos t \cos x \quad \text{on} \quad y = 0. \tag{18}$$

Next we differentiate (17) with respect to t and eliminate η^1_t from (17) and (18). This elimination yields

$$\phi^1_y + \omega_0^2 \phi^1_{tt} = \tfrac{1}{4}(3\omega_0^3 + \omega_0^{-1}) \sin 2t + \tfrac{3}{4}(\omega_0^3 - \omega_0^{-1}) \sin 2t \cos 2x$$
$$+ 2\omega_1 \cos t \cos x \quad \text{on} \quad y = 0. \tag{19}$$

To determine ϕ^1 we expand it in a Fourier cosine series since by (4), $\phi^1_x = 0$ at $x = 0$ and $x = \pi$. In order that ϕ^1 satisfy (1) and (4), we find that it must be of the form

$$\phi^1(x,y,t) = \sum_{n=0}^{\infty} A_n(t) \cos nx \cosh n(y+h). \tag{20}$$

Insertion of (20) into (19) yields

$$\omega_0^2 A_{0_{tt}} = \tfrac{1}{4}(3\omega_0^3 + \omega_0^{-1}) \sin 2t, \tag{21}$$

$$\omega_0^2 \cosh h \, . \, A_{1_{tt}} + \sinh h \, . \, A_1 = 2\omega_1 \cos t, \tag{22}$$

$$\omega_0^2 \cosh 2h \, . \, A_{2_{tt}} + 2 \sinh 2h \, . \, A_2 = \tfrac{3}{4}(\omega_0^3 - \omega_0^{-1}) \sin 2t, \tag{23}$$

$$\omega_0^2 \cosh nh \, . \, A_{n_{tt}} + n \sinh nh \, . \, A_n = 0 \quad (n = 3, 4, \ldots). \tag{24}$$

From (6) and (20) we see that all A_n with $n \geqslant 1$ must be periodic in t with period 2π. Then from (10) and (24) it follows that $A_n = 0$ for $n \geqslant 3$. From (10) and (23), we have

$$A_2 = -\frac{3}{16 \cosh 2h}(\omega_0 - \omega_0^{-7}) \sin 2t. \tag{25}$$

The periodicity of A_1 requires $\omega_1 = 0$, and (22) yields

$$A_1 = \alpha_1 \sin t + \beta_1 \cos t. \tag{26}$$

Here α_1 and β_1 are constants so far undetermined. Finally (21) yields

$$A_0 = -\tfrac{1}{16}(3\omega_0 + \omega_0^{-3}) \sin 2t + \alpha_0 t + \beta_0. \tag{27}$$

Here α_0 and β_0 are also undetermined constants.

We now insert the above results into (20), use the resulting expression for ϕ^1 in (17), and solve (17) for η^1. This yields

$$\eta^1 = \tfrac{1}{8}(\omega_0^2 - \omega_0^{-2}) - \omega_0\alpha_0 - \omega_0(\alpha_1 \cos t - \beta_1 \sin t)\cos x \cosh h + \tfrac{1}{8}(\omega_0^2 + \omega_0^{-2})\cos 2x + \tfrac{1}{8}(\omega_0^{-2} - 3\omega_0^{-6})\cos 2t \cos 2x. \quad (28)$$

By applying (7) to η^1 we find $\alpha_1 = 0$. Application of (8^1) to η^1 yields $\beta_1 = 0$. Utilization of (5) leads to

$$\alpha_0 = \tfrac{1}{8}(\omega_0 - \omega_0^{-3}). \quad (29)$$

Now that all constants except the inconsequential β_0 have been determined, the solution η^1, ϕ^1 and ω_1 is completely determined. It is given by

$$\eta^1 = \tfrac{1}{8}[(\omega_0^2 - \omega_0^{-2}) + (\omega_0^{-2} - 3\omega_0^{-6})\cos 2t]\cos 2x, \quad (30)$$

$$\phi^1 = \beta_0 + \tfrac{1}{8}(\omega_0 - \omega_0^{-3})t - \tfrac{1}{16}(3\omega_0 + \omega_0^{-3})\sin 2t - \frac{3}{16\cosh 2h}(\omega_0 - \omega_0^{-7}) \times \sin 2t \cos 2x \cosh 2(y+h), \quad (31)$$

$$\omega_1 = 0. \quad (32)$$

It is interesting to observe that η^1 contains a non-constant term which is independent of t and that ϕ^1 contains t-dependent terms independent of x and y, one of which is not periodic in t.

Let us now assume that η, ϕ and ω have second derivatives with respect to ϵ at $\epsilon = 0$, and let us denote them by η^2, ϕ^2 and ω_2. To determine them we differentiate equations (1) to (8) twice with respect to ϵ and set $\epsilon = 0$. Equations (1) and (4) to (7) remain unchanged in form while (8) takes the form of (8^1), with the superscript 'two' affixed to η and ϕ. The twice differentiated form of (2) and (3) is most easily found by differentiating (15) and (16). To solve the resulting equations we proceed exactly as before. Upon eliminating η^2 from (2^2) and (3^2), we obtain

$$\phi_y^2 + \omega_0^2\phi_{tt}^2 = \alpha_{11}\cos t \cos x + \alpha_{13}\cos t \cos 3x + \alpha_{31}\cos 3t \cos x + \alpha_{33}\cos 3t \cos 3x. \quad (33)$$

Here the constants α_{ij} are given by

$$\left.\begin{aligned} \alpha_{11} &= 2\omega_2 + \tfrac{1}{16}(-9\omega_0^{-7} + 12\omega_0^{-3} + 3\omega_0 + 2\omega_0^5), \\ \alpha_{13} &= \tfrac{1}{16}(\omega_0^{-7} - 5\omega_0 + 2\omega_0^5), \\ \alpha_{31} &= \tfrac{1}{16}(-9\omega_0^{-7} - 62\omega_0^{-3} + 31\omega_0), \\ \alpha_{33} &= \tfrac{3}{16}(9\omega_0^{-7} - 22\omega_0^{-3} + 13\omega_0). \end{aligned}\right\} \quad (34)$$

In solving for ϕ^2 as before we find that α_{11}, the coefficient of $\cos t \cos x$, must vanish. This yields for ω_2 the result

$$\omega_2 = \tfrac{1}{32}(9\omega_0^{-7} - 12\omega_0^{-3} - 3\omega_0 - 2\omega_0^5). \quad (35)$$

Then calculations like those previously given lead to the following solutions for η^2 and ϕ^2

$$\eta^2 = b_{11}\sin t \cos x + b_{13}\sin t \cos 3x + b_{31}\sin 3t \cos x + b_{33}\sin 3t \cos 3x, \quad (36)$$

$$\phi^2 = \beta_2 + \beta_{13}\cos t \cos 3x \cosh 3(y+h) + \beta_{31}\cos 3t \cos x \cosh(y+h) + \beta_{33}\cos 3t \cos 3x \cosh 3(y+h). \quad (37)$$

Here β_2 is arbitrary and the other constants are given by

$$\left.\begin{aligned} b_{11} &= \tfrac{1}{32}(3\omega_0^{-8}+6\omega_0^{-4}-5+2\omega_0^4), \\ b_{13} &= \tfrac{3}{128}(9\omega_0^{-8}+27\omega_0^{-4}-15+\omega_0^4+2\omega_0^8), \\ b_{31} &= \tfrac{1}{128}(3\omega_0^{-8}+18\omega_0^{-4}-5), \\ b_{33} &= \tfrac{3}{128}(-9\omega_0^{-12}+3\omega_0^{-8}-3\omega_0^{-4}+1), \end{aligned}\right\} \tag{38}$$

and

$$\left.\begin{aligned} \beta_{13} &= \frac{1}{128\cosh 3h}(1+3\omega_0^4)(3\omega_0^{-9}-5\omega_0^{-1}+2\omega_0^3), \\ \beta_{31} &= \frac{1}{128\cosh h}(9\omega_0^{-9}+62\omega_0^{-5}-31\omega_0^{-1}), \\ \beta_{33} &= \frac{1}{128\cosh 3h}(1+3\omega_0^4)(-9\omega_0^{-13}+22\omega_0^{-9}-13\omega_0^{-5}). \end{aligned}\right\} \tag{39}$$

4. Discussion

The finite Taylor expansions of $\epsilon\eta$, $\epsilon\phi$ and ω are

$$\epsilon\eta = \epsilon\eta^0(x,t)+\epsilon^2\eta^1(x,t)+\tfrac{1}{2}\epsilon^3\eta^2(x,t)+O(\epsilon^4), \tag{40}$$

$$\epsilon\phi = \epsilon\phi^0(x,y,t)+\epsilon^2\phi^1(x,y,t)+\tfrac{1}{2}\epsilon^3\phi^2(x,y,t)+O(\epsilon^4), \tag{41}$$

$$\omega = \omega_0+\tfrac{1}{2}\epsilon^2\omega_2+O(\epsilon^3). \tag{42}$$

The zero-order solution is given by equations (11) to (13), the first-order solution by (30) to (32) and the second-order solution by (35) to (37).

On the basis of (42) we see that the frequency ω depends upon the 'amplitude' ϵ of the wave motion, as is usually the case for non-linear systems. By examining (35) we find that $\omega_2 = 0$ at $\omega_0 \doteqdot 0{\cdot}89$ which corresponds to $h = h^* \doteqdot 1{\cdot}07$. For $h > h^*$, $\omega_2 < 0$ while for $h < h^*$, $\omega_2 > 0$. Thus, for depths greater than h^*, the fluid behaves like a soft spring, its free vibration frequency decreasing with increasing amplitude. For depths shallower than h^*, the fluid behaves like a hard spring, its free vibration frequency increasing with increasing amplitude.

From (40) and the expressions for η^0, η^1 and η^2, we see that the surface is never flat. It is most nearly so when $t = n\pi$ where n is any integer. Then

$$\epsilon\eta(x,n\pi) = \tfrac{1}{8}\epsilon^2(\omega_0^2+2\omega_0^{-2}-3\omega_0^{-6})\cos 2x. \tag{43}$$

The velocity vanishes throughout the fluid at $t = (n+\frac{1}{2})\pi$. At these times each part of the surface is either at its highest or lowest position. In particular, when n is an even integer, the crest is at $x = 0$ and the profile is given by

$$\begin{aligned} \epsilon\eta = \left[\epsilon+\frac{\epsilon^3}{256}(9\omega_0^{-8}+6\omega_0^{-4}-15+8\omega_0^4)\right]\cos x+\tfrac{1}{8}\epsilon^2(\omega_0^{-2}+3\omega_0^{-6})\cos 2x \\ +\tfrac{3}{256}\epsilon^3(9\omega_0^{-12}+6\omega_0^{-8}+30\omega_0^{-4}-16+\omega_0^4+2\omega_0^8)\cos 3x. \end{aligned} \tag{44}$$

The greatest rise occurs at $x = 0$, and it is equal to

$$\epsilon\eta_{\max} = \epsilon+\tfrac{1}{8}\epsilon^2(\omega_0^{-2}+3\omega_0^{-6})+\frac{\epsilon^3}{256}(27\omega_0^{-12}+27\omega_0^{-8}+96\omega_0^{-6}-63+11\omega_0^4+6\omega_0^8). \tag{45}$$

When $t = (n+\frac{1}{2})\pi$, with n an odd integer, the crest is at $x = \pi$. The wave profile at this time may be obtained either by reflecting that given by (44) about $x = \frac{1}{2}\pi$, or replacing ϵ by $-\epsilon$ in (44). A graph of the surface profile for $\epsilon = 0{\cdot}05$ and $h = 0{\cdot}25$ is shown in figure 1.

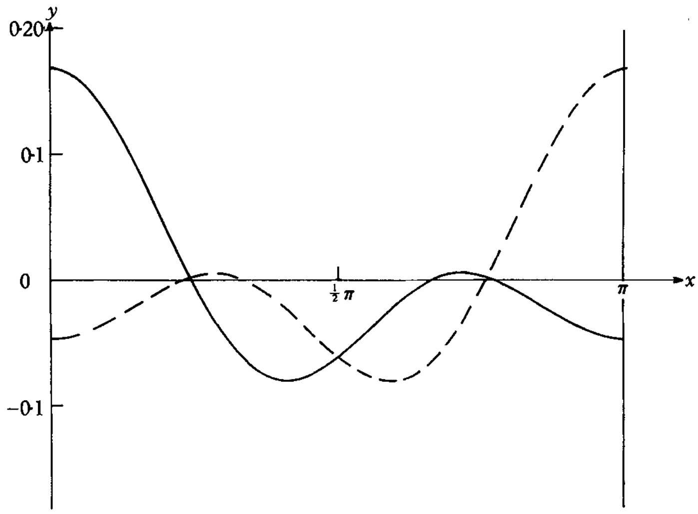

FIGURE 1. Profile of the surface of the standing wave at $t = (n+\frac{1}{2})\pi$, for n even (solid curve) and n odd (broken curve). These curves are based on equation (44) with $\epsilon = 0{\cdot}05$ and $h = 0{\cdot}25$.

When the depth h becomes infinite, our results (40) and (42) become

$$\epsilon\eta = (\epsilon+\tfrac{1}{32}\epsilon^3)\cos x+\tfrac{1}{2}\epsilon^2\cos 2x+\tfrac{3}{8}\epsilon^3\cos 3x, \tag{46}$$

$$\omega = 1-\tfrac{1}{8}\epsilon^2. \tag{47}$$

These results agree with those of Penney & Price who obtained additional terms in this case.

In calculating the pressure from (9) it is convenient to consider those instants when the fluid is at rest. Then in the vertical plane $x = 0$ and at the time of a crest, (9) becomes

$$\begin{aligned}\frac{k}{\rho g}(p-p_0) = &-y+\left[\epsilon+\frac{\epsilon^3}{256}(9\omega_0^{-8}-234\omega_0^{-4}+81-8\omega_0^4\right]\frac{\cosh(y+h)}{\cosh h}\\ &-\epsilon^2\left[\tfrac{1}{2}\omega_0^2+\tfrac{3}{8}(\omega_0^2-\omega_0^{-6})\frac{\cosh 2(y+h)}{\cosh 2h}\right]+\frac{\epsilon^3}{256}(1+3\omega_0^4)(27\omega_0^{-12}\\ &-63\omega_0^{-8}+39\omega_0^{-4}-5+2\omega_0^4)\frac{\cosh 3(y+h)}{\cosh 3h}\quad\text{for}\quad -h\leqslant y\leqslant\eta_{\max}.\end{aligned} \tag{48}$$

Upon replacing ϵ by $-\epsilon$ in (48), we obtain the pressure in the vertical plane $x = 0$ at the time of a trough. Graphs of these pressures are shown in figure 2 for $\epsilon = 0{\cdot}05$ and $h = 0{\cdot}25$.

When the depth h becomes infinite, our result (48) becomes

$$\frac{k}{\rho g}(p-p_0) = -y+(\epsilon-\tfrac{19}{32}\epsilon^3)e^y-\tfrac{1}{2}\epsilon^2. \tag{49}$$

This result agrees to the third order with the equation (135) of Penney & Price with the exception of the term independent of y. The difference arises from the fact that Penney & Price did not take into account the contribution of the zero order term in their potential function Φ. Had they done this by utilizing their equation (47), the correct form of their equation (135) to the fifth order would have been

$$p = \frac{\lambda\rho g}{2\pi}\left[-y-\tfrac{1}{2}A^2+\tfrac{3}{32}A^4+(A-\tfrac{19}{32}A^3+\tfrac{755}{1792}A^5)\,e^y - \tfrac{5}{14}A^4 e^{2y} - (\tfrac{7}{132}A^5)\,e^{3y} + (\tfrac{15}{256}A^5)\,e^{5y}\right].$$

Our solution describes the standing wave which results from the reflexion of a normally incident progressing wave from a wall or breakwater at $x = 0$. It also describes the free vibrations of the liquid contained between two vertical walls one-half wavelength apart. In either of these cases the pressure given by (48) and shown in figure 2 is the pressure on the wall.

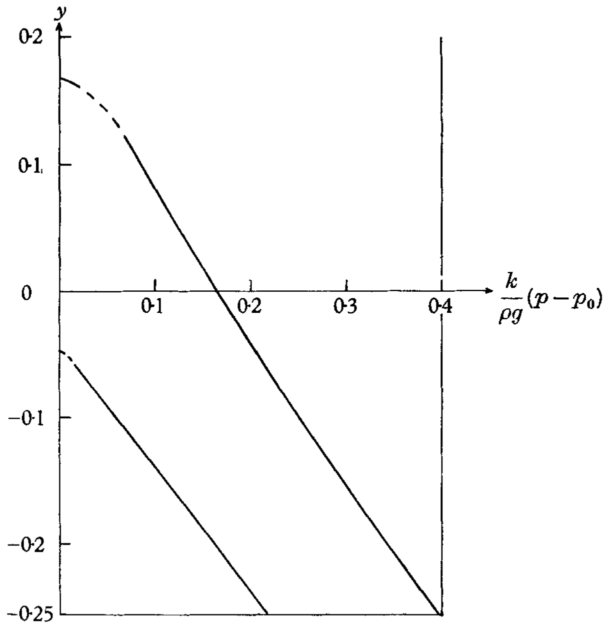

FIGURE 2. Pressure as a function of depth at $x = 0$ for $t = (n+\frac{1}{2})\pi$, n even (upper curve) and n odd (lower curve). These curves are based on equation (48) with $\epsilon = 0{\cdot}05$ and $h = 0{\cdot}255$. The horizontal scale is $k(p-p_0)/\rho g$ and the vertical scale is y. The dashed parts are sketched in to make $p-p_0$ have the value zero at the surface $y = \epsilon\eta_{\max}$ and $y = \epsilon\eta_{\min}$. Equation (48) does not yield $p-p_0 = 0$ at these points since it is correct only through ϵ^3.

Appendix

Following Penney & Price (1952), we write η and ϕ in a form which obviously satisfies (1), (4) and (5). It is

$$\epsilon\eta = \sum_{n=1}^{\infty} \alpha_n(t)\cos nx, \tag{1}$$

$$\epsilon\phi = \sum_{n=-\infty}^{\infty} \beta_n(t)\, e^{ny}\cos nx. \tag{2}$$

To satisfy (4) the coefficients $\beta_n(t)$ and $\beta_{-n}(t)$ must obey the relation

$$e^{nh}\beta_{-n}(t) = e^{-nh}\beta_n(t). \tag{3}$$

Upon inserting (1) and (2) into (2) and (3) of § 2, we obtain the following equations for the determination of α_n, β_n and ω:

$$\alpha_s = -\sum_{n=-\infty}^{\infty} \omega\beta_n'[E(n, s-n)+E(n, s+n)] - \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} mn$$
$$\times \beta_n\beta_m[E(m+n, s-m+n)+E(m+n, s+m-n)] \quad \text{for} \quad s = 0, 1, 2, \ldots, \tag{4}$$

$$\omega\alpha_s' = \sum_{n=-\infty}^{\infty} n\beta_n[E(n, s-n)+E(n, s+n)] - \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} mn\,\alpha_m\beta_n$$
$$\times [E(n, s-m+n)+E(n, s+m-n)-E(n, s-m-n)-E(n, s+m+n)]$$
$$\text{for} \quad s = 0, 1, 2, \ldots. \tag{5}$$

Here a prime denotes differentiation with respect to t and the quantity $E(\lambda, s)$ is defined by the equations

$$E(\lambda, \pm s) = \sum_{N=0}^{\infty} \frac{\lambda^N}{2^N N!} S_N(s), \tag{6}$$

$$S_N(\pm s) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} \ldots \alpha_m\alpha_n\alpha_p \ldots \alpha_{s-m-n-p}, \ldots, \tag{7}$$

$$\alpha_{-k} = \alpha_k \quad (k = 1, 2, 3, \ldots). \tag{8}$$

Instead of using the iteration method of Penney & Price, we solve (4) and (5) by assuming that α_n, β_n and ω can be expressed as

$$\epsilon\alpha_s = \sum_{j=s}^{\infty} \alpha_{sj}\epsilon^j, \tag{9}$$

$$\epsilon\beta_s = \sum_{j=|s|}^{\infty} \beta_{sj}\epsilon^j, \tag{10}$$

$$\omega = \sum_{j=0}^{\infty} \omega_j\epsilon^j. \tag{11}$$

Then $E(\lambda, s)$ can be written as

$$E(\lambda, s) = \sum_{j=|s|}^{\infty} E_j(\lambda, s)\,\epsilon^j. \tag{12}$$

We now insert equations (9) to (12) into (4) and (5) and equate the coefficients of the powers of ϵ. In this way we obtain, for each pair of integers $v \geqslant s \geqslant 0$, the equations

$$\alpha_{sv} = -\sum_{n=-v}^{v}\sum_{j=|n|}^{v}\sum_{k=0}^{j} \omega_k\beta_{n(j-k)}'[E_{v-j}(n, s-n)+E_{v-j}(n, s+n)]$$
$$-\frac{1}{2}\sum_{n=-v}^{v}\sum_{m=0}^{v}\sum_{j=m}^{v}\sum_{i=|n|}^{v} mn\,\alpha_{mj}\beta_{ni}[E_{v-i-j}(m+n, s-m+n)$$
$$+E_{v-i-j}(m+n, s+m-n)], \tag{13}$$

$$\sum_{k=0}^{v-s} \omega_k \alpha'_{s(v-k)} = \sum_{n=-v}^{v} \sum_{j=|n|}^{v} n\beta_{nj}[E_{v-j}(n, s-n) + E_{r-j}(n, s+n)]$$
$$-\frac{1}{2} \sum_{n=-v}^{v} \sum_{m=0}^{v} \sum_{j=m}^{v} \sum_{i=|n|}^{v} mn\, \alpha_{mj}\beta_{ni}[E_{v-i-j}(n, s-m+n) + E_{v-i-j}(n, s+m-n)$$
$$-E_{v-i-j}(n, s-m-n) - E_{v-i-j}(n, s+m+n)]. \quad (14)$$

These equations can be solved successively, starting with $s = v = 0$. We have solved them for all s and v satisfying $3 \geqslant v \geqslant s \geqslant 0$. The results are the same as those of § 3, but the labour is greater.

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